836 *KSME International Journal, Vol. 13, No. 11, pp. 836~841, 1999*

A Discrete-Particle Formulation of the Equations of Motion for Planar Electrodynamic Systems

Soo-Joon Kang* and Ki-Soon Park**

(Received June 1, 1999)

The fundamental properties of a planar electrodynamic system are investigated analytically. The electrically charged particle's motion is conservative and governed by a set of nonlinear ordinary differential equations. Linearization about static equilibrium admits normal mode behavior. As examples, the dynamics of a fixed-charge electroscope and an electrostatic string are considered. This paper suggests that a surface charge, when appropriately controlled, can be turned from a liability into an asset.

Key Words: Planar Electrodynamic, Electrically Charged Particle, Fixed-Charge Electroscope, Electrostatic String

1. Introduction

The limitations in modifying a structure's gross performance (mass, size, damping, stiffness, slewing rates etc.) by tailoring its material properties (fiber reinforcements, braids, weaves, energy dissipating parts, etc.) has led to the demand for performance modifications using active components (pneumatic and hydraulic actuators, piezoelectric sensors and actuators, imbedded fiber-optic sensors, motors, etc.) (Park, 1992; Bryant et al., 1986; Quinn, 1985; Lewis, 1985; Udd, 1985). Fundamentally, the forces that modify material properties and hence performance are internal electrical forces representing bonds between molecules. Within this context, design consists of judiciously arranging the molecules to yield the desired properties.

Topologically, these arrangements consist of molecules forming materials, and constituting a structure with sensors, actuators, computational elements, etc. The synthesis and reorganization of these elements into active (smart, adaptive, intelligent) structures are currently the focus of numerous engineering investigations (Wada, 1989; Chong et al., 1990).

By adopting a discrete particles approach, we formulate the equations governing the planar electrodynamics of interconnected electrical charged particles. Then, as simple examples, this paper describes the electrodynamics of the fixed charge electroscope and the electrostatic string.

2. Equations of Motion

We consider a planar electrodynamics system of n point charges suspended by an insulated massless string as shown in Fig. 1. As shown, inertially fixed charges (represented by black dots) and suspended charges (represented by white dots) are enclosed in a grounded electrodynamic chamber. The position and velocity vectors of the point charge are expressed as functions of the independent coordinates

$$\vec{r}_i=\vec{r}_i(\theta_1,\ \theta_2,\ \cdots,\ \theta_m),\ \dot{\vec{r}}_i=\sum_{j=1}^{m}\frac{\partial\vec{r}_i}{\partial\theta_j}\dot{\theta}_j \quad (i=1,\ 2,\ \cdots,\ n) \tag{1}$$

in which $\vec{r}_i=x_i\hat{n}_1+y_i\hat{n}_2(i=1,\ 2,\ \cdots,\ n)$ denotes the position vector of the i-th point charge, and $\theta_j(j=1,\ 2,\ \cdots,\ m)$ denotes the j-th angular position relative to the $\hat{n}_1$ axis about the $\hat{n}_3$ axis, and overdots denote derivatives with respect to time.

The kinetic energy of the system becomes

* Department of Mechanical Engineering Republic of Korea Air Force Academy, 363-849, Korea.

** HQ, Republic of Korea Air Force, 320-919, Korea.

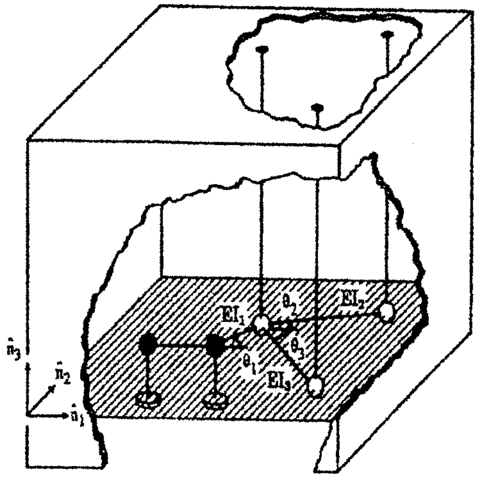


Fig. 1 Electrodynamic system.

$$T=\sum_{i=1}^{n}\frac{1}{2}m_i\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m}m_i\frac{\partial\vec{r}_i}{\partial\theta_j}\cdot\frac{\partial\vec{r}_i}{\partial\theta_k}\dot{\theta}_j\dot{\theta}_k \quad (2)$$

where m_i $(1, 2, \cdots, n)$ denotes the mass of the i-th point charge. We assume that the plane of motion is perpendicular to the gravity vector as indicated by the shaded region in Fig. 1.

The potential energy of the system is composed of the electrical potential U_e and gravitational potential U_g:

$$U=U_e+U_g \quad (3)$$

in which $U_e=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi q_i q_j}{|\vec{r}_i-\vec{r}_j|}$, $U_g=\frac{1}{2}\sum_{i=1}^{n}\frac{m_i g}{L}|\vec{r}_i-\vec{r}_i^{\,1}|^2$ (4)

where χ and $q_i (i=1, 2, \cdots, n)$ denote the electrical constant of the medium (Tipler, 1982) and fixed electrical point charges, respectively. The magnitude of the gravitational acceleration is denoted by g, the length of the massless suspended strings is denoted by L, and the nominal positions of the free ends of the suspended strings are denoted by $\vec{r}_i^{\,1}=x_i^1\hat{n}_1+y_i^1\hat{n}_2 (i=1, 2, \cdots, n)$. The fixed ends of the suspended strings are then located at $\vec{r}_i^{\,1}+L\hat{n}_3$.

Invoking Lagrange's equations of motion for conservative systems (Meirovitch, 1970), we obtain a set of nonlinear ordinary differential equations of the form

$$0=\frac{d}{dt}\left[\frac{\partial T}{\partial\dot{\theta}_l}\right]-\frac{\partial T}{\partial\theta_l}+\frac{\partial U}{\partial\theta_l}, \quad (l=1, 2, \cdots, m) \quad (5)$$

Substituting Eqs. (2) and (3) into (5) and carrying out the necessary differentiations yields the following set of equations governing the motion of the planar electrodynamic system:

$$\begin{aligned}0=&\sum_{i=1}^{n}\sum_{j=1}^{m}m_i\frac{\partial\vec{r}_i}{\partial\theta_k}\left[\frac{\partial\vec{r}_i}{\partial\theta_j}\ddot{\theta}_j+\frac{\partial^2\vec{r}_i}{\partial\theta_j^2}\dot{\theta}_j^2\right]\\&-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi q_i q_j}{|\vec{r}_i-\vec{r}_j|^2}\times\frac{\partial|\vec{r}_i-\vec{r}_j|}{\partial\theta_k}\\&+\sum_{i=1}^{n}\frac{m_i g}{L}|\vec{r}_i-\vec{r}_i^{\,1}|\cdot\frac{\partial}{\partial\theta_k}|\vec{r}_i-\vec{r}_i^{\,1}|\\&(k=1, 2, \cdots, m)\end{aligned} \quad (6)$$

3. Static Equilibrium

The static equilibrium positions are found by letting $\ddot{\theta}_k(t)=\dot{\theta}_k(t)=0$, $(k=1, 2, \cdots, m)$ in Eq. (6). We obtain the nonlinear set of algebraic equations

$$\begin{aligned}\frac{\partial U}{\partial\theta_k^o}=&-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\chi q_i q_j}{|\vec{r}_i^{\,o}-\vec{r}_j^{\,o}|^2}\cdot\frac{\partial}{\partial\theta_k^o}|\vec{r}_i^{\,o}-\vec{r}_j^{\,o}|\\&+\sum_{i=1}^{n}\frac{m_i g}{L}|\vec{r}_i^{\,o}-\vec{r}_i^{\,1}|\cdot\frac{\partial}{\partial\theta_k^o}|\vec{r}_i^{\,o}-\vec{r}_i^{\,1}|\\=&0, \ (k=1, 2, 3, \cdots, m)\end{aligned} \quad (7)$$

in which $\vec{r}_i^{\,o}$ $(i=1, 2, \cdots, n)$ and $\theta_j^o (j=1, 2, \cdots, m)$ denote the equilibrium positions.

Our interest lies now in determining charges that yield a desirable equilibrium position. Eq. (7) represents a set of m quadratically nonlinear algebraic equations expressed in terms of unknown charges $q_i (i=1, 2, \cdots, n)$. Equation (7) is written as

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ijk}q_i q_j=d_k, \quad (k=1, 2, \cdots, m) \quad (8)$$

in which

$$a_{ijk}=\frac{\chi}{|\vec{r}_i^{\,o}-\vec{r}_j^{\,o}|^2}\cdot\frac{\partial}{\partial\theta_k^o}|\vec{r}_i^{\,o}-\vec{r}_j^{\,o}|,$$
$$d_k=\sum_{i=1}^{n}\frac{m_i g}{L}|\vec{r}_i^{\,o}-\vec{r}_i^{\,1}|\cdot\frac{\partial}{\partial\theta_k^o}|\vec{r}_i^{\,o}-\vec{r}_i^{\,1}| \quad (9)$$

The exact solution to Eq. (8) can be obtained numerically. As an alternative, an approximate solution can be obtained by a first-order perturbation analysis (Meirovitch, 1970), in which we let

$$a_{ijk}=a_{ijk}^o+a_{ijk}^1, \quad q_i=q_i^o\ q_i^1 \quad (10)$$

where a_{ijk}^o and q_i^o are nominal quantities and a_{ijk}^1 and q_i^1 are perturbations. Substituting Eq. (10) into Eq. (8) and neglecting second-order terms yields the following set of linear algebraic equations:

$$\sum_{j=1}^{n} b_{ij} q_j^1 = c_i, \ (i=1, 2, \cdots, m) \tag{11}$$

where

$$b_{ij} = 2\sum_{k=1}^{n} a_{jki}^o q_k^o, \ c_i = -\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jki}^1 q_j^o q_k^o \tag{12}$$

in which b_{ij} and c_i are known coefficients and unknown perturbations in the electrical charges, respectively. The solution to Eq. (11) yields q_j^1, $(j=1, 2, \cdots, n)$.

4. Linearization

The linearized electrodynamics relative to equilibrium are now considered by substituting the relative angular positions $\eta_i(t) = \theta_i(t) - \theta_i^o$ $(i = 1, 2, \cdots, m)$ into Eqs. (6) and (7), and expressing them in the functional forms

$$f_i(\eta_1, \eta_2, \cdots, \eta_m, \dot{\eta}_1, \dot{\eta}_2, \cdots, \dot{\eta}_m, \ddot{\eta}_1, \ddot{\eta}_2, \cdots, \ddot{\eta}_m) = 0, \ (i=1, 2, \cdots, m) \tag{13}$$

and

$$f_i(0, 0, \cdots, 0) = 0, \ (i=1, 2, \cdots, m). \tag{14}$$

respectively. Taylor series approximations of Eq. (13) about static equilibrium yield the following linearized equations of motion for the electrodynamic system:

$$\sum_{j=1}^{m} (m_{ij}\ddot{\eta}_j + k_{ij}\eta_j) = 0, \ (i=1, 2, \cdots, m) \tag{15}$$

where

$$m_{ij} = m_{ji} = \left(\frac{\partial f_i}{\partial \theta_j}\right)^0 = \sum_{k=1}^{n} m_k \frac{\partial \vec{r}_k^o}{\partial \vec{r}_i^o} \cdot \frac{\partial \vec{r}_k^o}{\partial \vec{r}_j^o} \tag{16}$$

and

$$\begin{aligned} k_{ij} = k_{ji} &= \left(\frac{\partial f_i}{\partial \theta_j}\right)^0 = \left(\frac{\partial^2 U}{\partial \theta_j \, \partial \theta_i}\right)^0 \\ &= \sum_{k=1}^{n-1}\sum_{l=k+1}^{n} \frac{\chi q_k q_l}{|\vec{r}_k^o - \vec{r}_l^o|^3} \cdot \left[\frac{\partial^2 |\vec{r}_k^o - \vec{r}_l^o|}{\partial \theta_i^{o2}} \cdot \delta_{ij} |\vec{r}_k^o - \vec{r}_l^o| \right. \\ &\quad \left. - 2\frac{\partial |\vec{r}_k^o - \vec{r}_l^o|}{\partial \theta_i^o} \cdot \frac{|\vec{r}_k^o - \vec{r}_l^o|}{\partial \theta_l^o}\right] \\ &\quad + \left(\sum_{k=1}^{n} \frac{m_k g}{2L} \cdot \frac{\partial^2}{\partial \theta_j^{o2}} |\vec{r}_k^o - \vec{r}_k^1|^2\right)\delta_{ij} \end{aligned} \tag{17}$$

in which m_{ij} are elements of a positive definite and symmetric inertia matrix, k_{ij} are elements of a positive definite and symmetric stiffness matrix, and δ_{ij} denotes the Kronecker delta function. These symmetry and definiteness properties imply that Eq. (15) admits normal mode behavior (Meirovitch, 1971). An associated set of m real modes of vibration are then governed by the eigenvalue problem

$$(\omega^{(k)})^2 \sum_{j=1}^{m} m_{ij}\phi_j^{(k)} = \sum_{j=1}^{m} k_{ij}\phi_j^{(k)}, \ (i, k=1, 2, \cdots, m) \tag{18}$$

in which $\phi_j^{(k)}$ $(j=1, 2, \cdots, m)$ denotes the k-th natural mode of vibration, and $\omega^{(k)}$ denotes the associated k-th natural frequency of oscillation. The natural modes of vibration are mutually orthogonal and satisfy the orthonormality conditions

$$\begin{aligned} \phi_i^{(r)} m_{ij} \phi_j^{(s)} = \delta_{rs}, \ &\sum_{i=1}^{m}\sum_{j=1}^{m} \phi_i^{(r)} m_{ij} \phi_j^{(s)} \\ &= (w^{(r)})^2 \delta_{rs}, \ (r, s=1, 2, \cdots, m). \end{aligned} \tag{19}$$

5. Numerical Examples

5.1 Fixed charge electroscope

The position vectors in Eq. (1) are given by (see Fig. 2)

$$\begin{aligned} &\vec{r}_1 = -d_o \hat{i}, \ \vec{r}_2 = 0, \ \vec{r}_3 = d(\cos\theta_1 \hat{i} + \sin\theta_1 \hat{j}), \\ &\vec{r}_4 = d(\cos\theta_2 \hat{i} - \sin\theta_2 \hat{j}) \end{aligned} \tag{20}$$

Neglecting gravitational effects by letting $g=0$, we obtain from Eq. (6) the equations of motion governing the fixed charge electroscope as follows:

$$0 = m_3 d^2 \ddot{\theta}_1 + \frac{\chi q_1 q_3}{[(d_0 + dc_1)^2 + (ds_1)^2]^{\frac{3}{2}}} d_0 d s_1$$

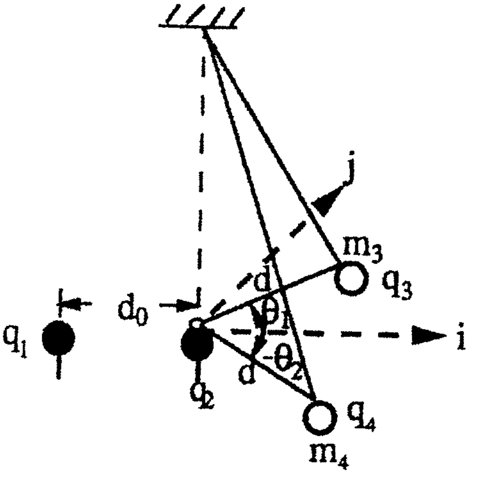


Fig. 2 Fixed charge electroscope.

$$-\frac{\chi q_3 q_4}{[d^2(c_1-c_2)^2+d^2(s_1+s_2)^2]^{\frac{3}{2}}}\times d^2(c_2s_1+s_2c_1) \quad (21)$$

$$0=m_4d^2\ddot{\theta}_2+\frac{\chi q_1 q_4}{[(d_0+dc_2)^2+(ds_2)^2]^{\frac{3}{2}}}d_0ds_1$$

$$-\frac{\chi q_3 q_4}{[d^2(c_1-c_2)^2+d^2(s_1+s_2)^2]^{\frac{3}{2}}}\times d^2(c_2s_1+s_2c_1) \quad (22)$$

where $c_1\equiv\cos\theta_1$, $c_2\equiv\cos\theta_2$, $s_1\equiv\sin\theta_1$ and $s_2\equiv\sin\theta_2$. Let us now nondimensionalize the system by introducing the independent parameters $p_0=d_0/d$, $p_1=m_3/m_4$, $p_2=q_2/q_1$, $p_3=q_3/q_1$, $p_4=q_4/q_1$, and $p_5=d^3m_4/\chi q_1^2$, where $\chi=9\times10^9(N\cdot m^2/Coulomb^2)$.

Introducing the independent parameters into the equations of motion, Eqs. (21) and (22), yields

$$0=p_1p_5\ddot{\theta}_1+\frac{p_0p_3}{[p_0^2+2p_0c_1+1]^{\frac{3}{2}}}s_1$$

$$-\frac{p_3p_4}{[2-2c_1c_2+2s_1s_2]^{\frac{3}{2}}}(c_2s_1+s_2c_1) \quad (23)$$

$$0=p_5\ddot{\theta}_2+\frac{p_0p_4}{[p_0^2+2p_0c_2+1]^{\frac{3}{2}}}s_2$$

$$-\frac{p_3p_4}{[2-2c_1c_2+2s_1s_2]^{\frac{3}{2}}}(c_2s_1+s_2c_1), \quad (24)$$

By letting $\ddot{\theta}_1=\ddot{\theta}_2=0$ in Eqs. (23) and (24), the static equilibrium position of the fixed charge electroscope satisfies

$$0=\frac{p_o\cdot s_1^o}{(p_o^2+2p_oc_1^o+1)^{3/2}}-\frac{p_4\cdot(c_2^os_1^o+s_2^oc_1^o)}{(2-2c_1^oc_2^o+2s_1^os_2^o)^{3/2}} \quad (25)$$

$$0=\frac{p_o\cdot s_2^o}{(p_o^2+2p_oc_1^o+1)^{3/2}}-\frac{p_3\cdot(c_2^os_1^o+s_2^oc_1^o)}{(2-2c_1^oc_2^o+2s_1^os_2^o)^{3/2}} \quad (26)$$

Figure 3 shows the equilibrium angles of the fixed charge electroscope in terms of the independent dimensionless parameters, q_3/q_4 and d_0/d, varied from 0.2 to 2.0 for $p_1=p_2=p_4=d=q_4=m_4=1$.

The linearized differential equations governing the motion of the fixed charge electroscope are given by Eq. (15), in which

$$m_{11}=p_1p_5,\ m_{12}=m_{21}=0,\ m_{22}=p_5 \quad (27)$$

and

$$k_{11}=\frac{p_0p_3c_1[p_0^2+2p_0c_1+1]+3p_0^2p_3s_1^2}{[p_0^2+2p_0c_1+1]^{\frac{5}{2}}}$$

$$-p_3p_4\times\left[\frac{(c_1c_2-s_1s_2)(2-2c_1c_2+2s_1s_2)}{[2-2c_1c_2+2s_1s_2]^{\frac{5}{2}}}\right.$$

$$\left.-\frac{3(c_1s_2+c_2s_1)^2}{[2-2c_1c_2+2s_1s_2]^{\frac{5}{2}}}\right] \quad (28)$$

$$k_{12}=k_{21}=-p_3p_4\times\left[\frac{(c_1c_2-s_1s_2)(2-2c_1c_2+2s_1s_2)}{[2-2c_1c_2+2s_1s_2]^{\frac{5}{2}}}\right.$$

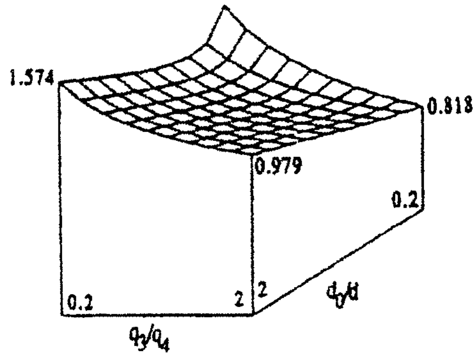
$$\left.-\frac{3(c_1s_2+c_2s_1)^2}{[2-2c_1c_2+2s_1s_2]^{\frac{5}{2}}}\right] \quad (29)$$

$$k_{22}=\frac{p_0p_3c_2[p_0^2+2p_0c_2+1]+3p_0^2p_3s_2^2}{[p_0^2+2p_0c_1+1]^{\frac{5}{2}}}$$

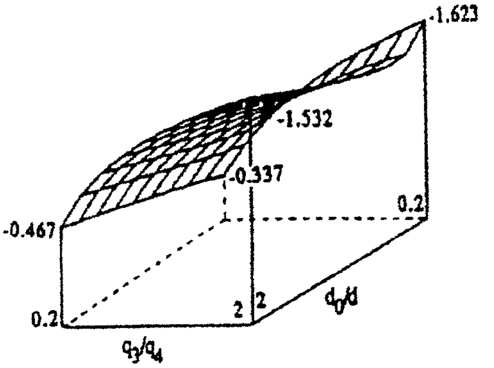
$$-p_3p_4\times\left[\frac{(c_1c_2-s_1s_2)(2-2c_1c_2+2s_1s_2)}{[2-2c_1c_2+2s_1s_2]^{\frac{5}{2}}}\right.$$

$$\left.-\frac{3(c_1s_2+c_2s_1)^2}{[2-2c_1c_2+2s_1s_2]^{\frac{5}{2}}}\right] \quad (30)$$

where $c_1\equiv\cos\theta_1$, $c_2\equiv\cos\theta_2$, $s_1\equiv\sin\theta_1$ and $s_2\equiv\sin\theta_2$.



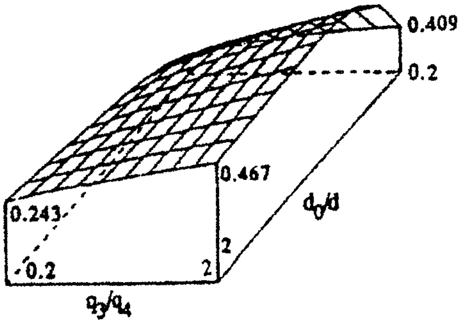
Angle 1



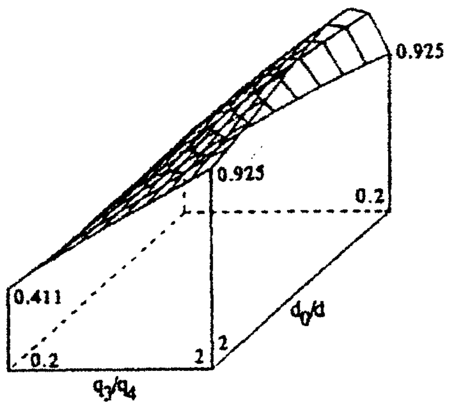
Angle 2

Fig. 3 Static equilibrium positions of the fixed charge electroscope.

The natural modes of vibration and the associated natural frequencies of oscillation are shown in Figs. 4 and 5. The two natural modes of vibration, denoted by $\phi_1=(\phi_1^{(1)},\ \phi_2^{(1)})$ and $\phi_2=$



Natural frequency 1



Natural frequency 1

Fig. 4 Natural frequencies of the fixed charge electroscope.

$(\phi_1^{(2)}, \phi_2^{(2)})$, are normalized such that $\phi_2^{(1)} = \phi_2^{(2)} = 1$.

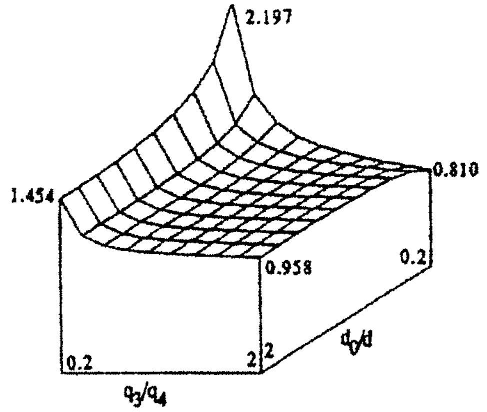
5.2 Electrostatic String

The position vectors in Eq. (1) are given by (see Fig. 6)

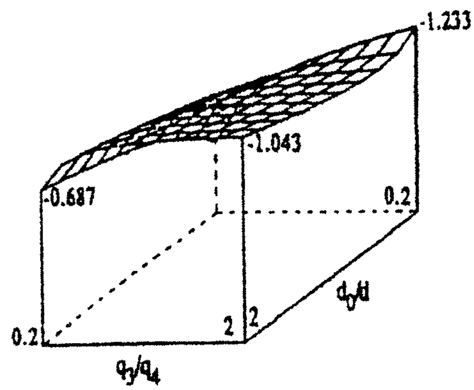
$$\vec{r}_1 = 0\,\hat{i},\ \vec{r}_2 = d_i\,\hat{i},\ \vec{r}_k = \overrightarrow{r_{k-1}} + d(\cos\theta_{k-2}\,\hat{i} + \sin\theta_{k-2}\,\hat{j}),\ (k = 3, 4, \cdots, 10) \qquad (31)$$

where d is the constant length of a rigidly pinned connection.

The equilibrium position of the electrostatic string is $\theta_1^o = \theta_2^o = \ldots = \theta_8^o = 0$. Linearization of the differential equations of motion leads to the natural modes of vibration and the associated natural frequencies of oscillation as shown in Fig. 7. The results are for $q_i = 1$ $(i = 1, 2, \cdots, 10)$, $d = 1$, and $m = 1$.



Mode 1 $(\phi_1^{(1)})$



Mode 2 $(\phi_1^{(2)})$

Fig. 5 Associated natural modes of the fixed charge electroscope.

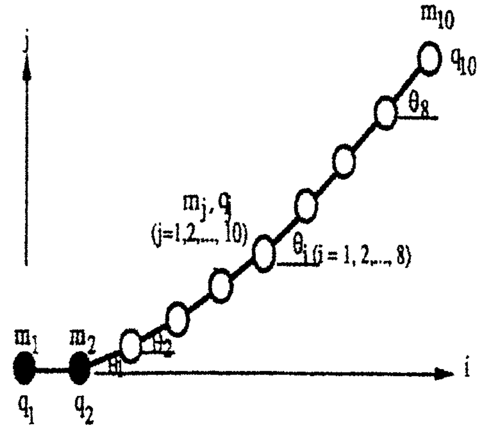


Fig. 6 Electrostatic string.

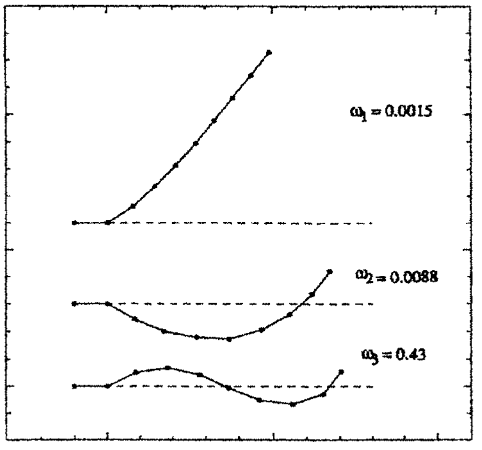


Fig. 7 Lowest 3 natural modes and frequencies of the electrostatic string.

6. Conclusions

The fundamental properties of planar electrodynamic systems, whose dynamic characteristics are predominantly derived from external free electron forces, are investigated analytically. The demonstrated ability to predict the behavior of simple electrodynamic systems has important implications. The dominance of the linearity is attractive under proper conditions. It implies that the analysis and design strategies of electrodynamic systems can be readily carried out on the basis of linear theories for electrodynamic systems having more complex geometries. But it is suggested that a verification is required by appropriate experimental work.

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